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First integrals, integrating factors and λ-symmetries of second-order differential equations

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Abstract
For a given second-order ordinary differential equation (ODE), several relationships among first integrals, integrating factors and λ-symmetries are studied. The knowledge of a λ-symmetry of the equation permits the determination of an integrating factor or a first integral by means of coupled first-order linear systems of partial differential equations. If two nonequivalent λ-symmetries of the equation are known, then an algorithm to find two functionally independent first integrals is provided. These methods include and complete other methods to find integrating factors or first integrals that are based on variational derivatives or in the Prelle–Singer method. These results are applied to several ODEs that appear in the study of relevant equations of mathematical physics.

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1. Introduction

First integrals and integrating factors play a central role in the study of ordinary differential equations (ODEs). In fact, finding a first integral of a given ODE is equivalent to obtaining an integrating factor of the equation. Several authors have obtained necessary and sufficient conditions for a function $\mu(x, u, \dot{u})$ to be an integrating factor of a second-order ODE $\ddot{u} = \phi(x, u, \dot{u})$. Most of their approaches rest on the fact that the function $\mu(\ddot{u} - \phi(x, u, \dot{u}))$ is a total derivative and therefore its variational derivative is null. As a consequence integrating factors can be determined as solutions of a second-order linear system of PDEs [1–4]. Since solving this system is usually a more difficult task than solving the original ODE, many studies have been done to investigate special classes of integrating factors, through specific *ansätze* for μ [2, 5]. In [6] and [2] integrating factors appear as the solutions of the adjoint equation of the linearized equation and an additional equation that describes an extra adjoint-invariance condition.

Integrating factors have also been connected with the symmetries of the ODE. For a first-order ODE, Lie's symmetry reduction yields the quadrature of the equation. Lie also showed that this is equivalent to finding a first integral and the corresponding integrating factor of the ODE. However, the equivalence between integrating factors and Lie point symmetries fails for higher-order equations, because there exist exact equations without Lie point symmetries [7, 8].

The integrability of equations that lack Lie point symmetries has been considered by several authors: nonlocal symmetries and hidden symmetries hold important roles in these studies [9–11]. However, the main problem associated with nonlocal symmetries is how to determine them. Although no general method to calculate them has been derived, several strategies can be followed in some cases [12]. A route to the determination of some nonlocal symmetries necessary to the complete specification of some nonlinear 1+1 evolution equations has been designed by Myeni and Leach [13, 14].

Many of these nonlocal or hidden symmetries can be connected with λ-symmetries (also called $\mathcal{C}^{\infty}$-symmetries) [15]. These λ-symmetries can be calculated by a well-defined algorithm, include Lie point symmetries as a very specific subclass, and have an associated order reduction procedure, somehow similar to the classical Lie method of reduction [16]. Although λ-symmetries are not Lie point symmetries, the unique prolongations of vector fields (in the space (x, u)) to the space of variables $(x, u, \ldots, u^{n)})$ for which the Lie reduction method applies are always λ-prolongations, for some functions $\lambda(x, u, \dot{u})$. This result has been proved by the authors [17] and is implicit in the work of Pucci and Sacomandi on telescopic vector fields [18]. From a geometrical point of view several studies and interpretations on λ-symmetries have been made by several authors [19–21] including further extensions of λ-symmetries to systems [22, 23], to PDEs [24] and to variational problems [25, 26].

In the context of integrating factors, two important facts are that any exact ODE admits a λ-symmetry, although the equation may have no Lie point symmetries, and that a first integral of the ODE appears in the reduction procedure associated with the λ-symmetry [27]. This motivates us to study in depth the relationships among first integrals, integrating factors and λ-symmetries, which is one of the aims of this paper. For simplicity we present here the results for second-order ODEs.

This paper is organized as follows. We first prove (theorem 2) that any first integral of a second-order ODE determines a λ-symmetry of the equation and, conversely, there is a class of first integrals associated with any given λ-symmetry. As a consequence a procedure to determine a first integral, when a λ-symmetry is known, is derived; only first-order ODEs are involved in this algorithm. This method extends the results in [7] and can be applied to the examples that appear therein; some of these examples correspond to ODEs without Lie point symmetries.

Once it has been shown how a first integral I determines an integrating factor and a λ-symmetry of the equation, section 3 is devoted to the study of conditions on two functions μ and λ in order for μ to be an integrating factor or $v = \partial u$ to be a λ-symmetry of the equation. This is done through the study of the compatibility of a first-order system which involves the two functions μ and λ and an unknown function I. The compatibility of the system permits the determination of a first integral of the equation through a line integral.

In section 4 it is shown that, when a λ-symmetry of the equation is known, a class of integrating factors of the equation can be determined as solutions of a first-order linear system of two coupled PDEs, which could be solved by standard methods. At this point it should be mentioned that the classical determining equations of the integrating factors of a second-order equation constitute a system of two second-order PDEs [1, 2]. In section 6 we derive these determining equations without using variational derivatives. As a particular case, when a Lie point symmetry of the equation is known, that first-order linear system can be expressed in terms of the characteristic of the symmetry and its solutions are integrating factors of the ODE. As far as we know, this is the first time that a direct connection between Lie point symmetries and integrating factors is given for equations of order higher than 1. A different and more indirect method to derive first integrals through Lie group analysis can be seen in [28].

It is well known that two independent first integrals of a second-order ODE provide the general solution of the equation. It is natural to raise the question whether two different λ-symmetries of the equation yield two independent first integrals. This is addressed in section 5 and an easy-to-check condition determines when two independent first integrals, associated with these two λ-symmetries, can be found.

These results are applied to the Ermakov–Pinney equation [29]. This equation has wide applications in several branches of physics, such as in the analysis of traveling-wave solutions associated with Schrödinger equations, accelerator physics or the one-dimensional Vlasov–Maxwell equations. It also arises in the moving shoreline analysis of rotating liquid motion in a circular paraboloidal basin. The Ermakov–Pinney equation has widely been studied in the context of symmetry analysis [30–33]. In this paper, as a direct consequence of our results, we present a novel derivation of the nonlinear superposition principle and its relationship with the Schrödinger equation.

In section 7 we first consider the relationship between our methods and the Prelle–Singer method of constructing integrating factors. This was introduced in [34] for first-order ODEs, has been adapted and applied to second-order ODEs in [35] and has been extended to nth-order ODEs in [36]. For a second-order equation of the form $\ddot{u} = \phi(x, u, \dot{u})$, the method tries to add to the differential form $\phi \mathrm{d}x - \mathrm{d}\dot{u}$ a *ghost* differential form $S(x, u, \dot{u})(\dot{u}\,\mathrm{d}x - \mathrm{d}u)$ in order for the resulting differential form to admit an integrating factor. We show that this happens if and only if $v = \partial_u$ is a λ-symmetry of the equation for $\lambda = -S$. In fact the examples given in [35] correspond to Lie symmetries of the equations. In this section we also consider the relationship between our methods and the method based on variational derivatives to obtaining integrating factors. This provides alternative proofs of some of our results.

Needless to say, examples have been chosen to illustrate easily the algorithms rather than to obtain innovative results on the equations. However, the generality of the issues will hopefully yield fruitful results when applied to a wide variety of physically important equations.

2. First integrals and λ-symmetries

This paper is devoted to investigating the relationships among first integrals, integrating factors and λ-symmetries of a given second-order ordinary differential equation

$$\ddot{u} = \phi(x, u, \dot{u}). \tag{1}$$

We denote by $A = \partial_x + \dot{u}\partial_u + \phi(x, u, \dot{u})\partial_{\dot{u}}$ the vector field associated with equation (1). In terms of A a first integral of (1) is any function $I(x, u, \dot{u})$ such that $A(I) = 0$.

An integrating factor of equation (1) is any function $\mu(x, u, \dot{u})$ such that

$$\mu[\ddot{u} - \phi(x, u, \dot{u})] = D_x I, \tag{2}$$

for some function $I(x, u, \dot{u})$, where D_x is the total derivative vector field: $D_x = \partial_x + \dot{u}\partial_u + \ddot{u}\partial_{\dot{u}} + \cdots$.

The relationship between first integrals and integrating factors is well known. We put this relationship in the next theorem for further reference.

Theorem 1.

(a) If $I(x, u, \dot{u})$ is a first integral of equation (1), then $\mu = I_{\dot{u}}$ is an integrating factor of (1).
(b) Conversely, if $\mu(x, u, \dot{u})$ is an integrating factor of (1), then there exists a first integral $I(x, u, \dot{u})$ of (1) such that $\mu = I_{\dot{u}}$.

Proof.

(a) If $AI = 0$, then $I_x + \dot{u} I_u + \phi(x, u, \dot{u}) I_{\dot{u}} = 0$. Therefore $I_x + \dot{u} I_u = -\phi(x, u, \dot{u}) I_{\dot{u}}$ and

$$D_x I = I_x + \dot{u} I_u + \ddot{u} I_{\dot{u}} = -\phi(x, u, \dot{u}) I_{\dot{u}} + \ddot{u} I_{\dot{u}} = I_{\dot{u}}[\ddot{u} - \phi(x, u, \dot{u})]. \tag{3}$$

(b) If $\mu[\ddot{u} - \phi(x, u, \dot{u})] = D_x I$, for some function $I(x, u, \dot{u})$ then necessarily $\mu = I_{\dot{u}}$ and $-\mu\phi = -I_{\dot{u}}\phi = I_x + \dot{u} I_u$. This proves that $I_x + \dot{u} I_u + \phi(x, u, \dot{u}) I_{\dot{u}} = 0$, i.e., $AI = 0$. □

If $v = \xi(x, u)\partial_x + \eta(x, u)\partial_u$ is a vector field and $\lambda = \lambda(x, u, \dot{u})$ is a smooth function on the space of variables $(x, u, \dot{u})$, then the nth-order λ-prolongation $v^{[\lambda,(n)]}$ of v [16] can be characterized as the unique vector field on the space $M^{(n)}$ of variables $(x, u, \dot{u}, \ldots, u^{n)})$ such that

$$[v^{[\lambda,(n)]}, D_x] = \lambda v^{[\lambda,(n)]} + \nu D_x, \tag{4}$$

where $\nu = -(D_x + \lambda)(\xi) = -(A + \lambda)(\xi)$. When $n = 1$, the explicit form of $v^{[\lambda,(1)]}$ is

$$v^{[\lambda,(1)]} = \xi\partial_x + \eta\partial_u + [(A + \lambda)(\eta) - (A + \lambda)(\xi)\dot{u}]\partial_{\dot{u}}. \tag{5}$$

The vector field v is a λ-symmetry of equation (1) if and only if

$$[v^{[\lambda,(1)]}, A] = \lambda v^{[\lambda,(1)]} + \nu A. \tag{6}$$

When $v = \partial_u$, v is a λ-symmetry of (1) if and only if

$$\phi_u + \lambda\phi_{\dot{u}} = A(\lambda) + \lambda^2. \tag{7}$$

Our next theorem establishes a first relationship between first integrals and λ-symmetries.

Theorem 2.

(a) If $I(x, u, \dot{u})$ is a first integral of (1), then the vector field $v = \partial_u$ is a λ-symmetry of (1) for $\lambda = -I_u/I_{\dot{u}}$ and $v^{[\lambda,(1)]} I = 0$.
(b) Conversely, if $v = \partial_u$ is a λ-symmetry of (1) for some function $\lambda(x, u, \dot{u})$, then there exists a first integral $I(x, u, \dot{u})$ of (1) such that $v^{[\lambda,(1)]} I = 0$.

Proof.

(a) Since for any function $\lambda(x, u, \dot{u}) v^{[\lambda,(1)]} = \partial_u + \lambda\partial_{\dot{u}}$, it is clear that

$$v^{[\lambda,(1)]} I = I_u - \frac{I_u}{I_{\dot{u}}} I_{\dot{u}} = 0$$

when $\lambda = -I_u/I_{\dot{u}}$. Since $h(x, u, \dot{u}) = x$ and $I(x, u, \dot{u})$ are first integrals of $v^{[\lambda,(1)]}$, $D_x I$ is an invariant of $v^{[\lambda,(2)]}$ [16]. By applying $v^{[\lambda,(2)]}$ to both members of the identity $I_{\dot{u}}[\ddot{u} - \phi(x, u, \dot{u})] = D_x I$ we obtain

$$I_{\dot{u}} \cdot v^{[\lambda,(2)]}[\ddot{u} - \phi] + v^{[\lambda,(2)]}(I_{\dot{u}}) \cdot [\ddot{u} - \phi] = v^{[\lambda,(2)]}(D_x I) = 0.$$

This implies that

$$I_{\dot{u}} \cdot v^{[\lambda,(2)]}[\ddot{u} - \phi(x, u, \dot{u})] = 0 \qquad \text{when} \qquad \ddot{u} = \phi(x, u, \dot{u}).$$

Since $I_{\dot{u}} \neq 0$, it is clear that $v = \partial_u$ is a λ-symmetry of (1).

(b) If $v = \partial_u$ is a λ-symmetry of (1) for some function $\lambda(x, u, \dot{u})$, then

$$[v^{[\lambda,(1)]}, A] = \lambda \cdot v^{[\lambda,(1)]}. \tag{8}$$

Therefore $\{v^{[\lambda,(1)]}, A\}$ is an involutive set of vector fields in $M^{(1)}$ and there exists a function $I(x, u, \dot{u})$ such that $v^{[\lambda,(1)]}I = 0$ and $AI = 0$. □

Theorem 2 provides us a procedure to determine a first integral of (1) when we know that $v = \partial_u$ is a λ-symmetry for some known function $\lambda(x, u, \dot{u})$.

Suppose that $w(x, u, \dot{u})$ is a nontrivial first integral of $v^{[\lambda,(1)]}$, i.e., $w(x, u, \dot{u})$ is a solution of the first-order partial differential equation

$$w_u + \lambda(x, u, \dot{u}) w_{\dot{u}} = 0 \tag{9}$$

such that $w_{\dot{u}} \neq 0$. Since $h(x, u, \dot{u}) = x$ is obviously another first integral of $v^{[\lambda,(1)]}$, any first integral $I(x, u, \dot{u})$ of $v^{[\lambda,(1)]}$ can be written in the form

$$I(x, u, \dot{u}) = G(x, w(x, u, \dot{u})) \tag{10}$$

for some function of two variables $G(x, w)$.

By theorem 2 we can search for a common first integral for the vector fields $v^{[\lambda,(1)]}$ and A. We first observe that, if $I(x, u, \dot{u})$ has the form given in (10), then

$$\begin{aligned} A(I) &= I_x + \dot{u} I_u + \phi(x, u, \dot{u}) I_{\dot{u}} \\ &= (G_x + G_w w_x) + \dot{u}(G_w w_u) + \phi(x, u, \dot{u})(G_w w_{\dot{u}}) \\ &= G_x + G_w \cdot [w_x + \dot{u} w_u + \phi(x, u, \dot{u}) w_{\dot{u}}] = G_x + A(w) \cdot G_w. \end{aligned} \tag{11}$$

Hence in order to find a common first integral of the vector fields $v^{[\lambda,(1)]}$ and A we must solve the equation

$$G_x + A(w) \cdot G_w = 0. \tag{12}$$

In principle, equation (12) is a first-order partial differential equation in variables $(x, u, \dot{u})$. Nevertheless that equation can be considered as a first-order ordinary differential equation because $A(w)$ depends only on (x, w). This is a consequence of (8): since $v^{[\lambda,(1)]}(w) = 0$, we have that $v^{[\lambda,(1)]}(A(w)) = 0$ and $A(w)$ functionally depends on the first integrals $h(x, u, \dot{u}) = x$ and $w(x, u, \dot{u})$. Therefore there exists a function of two variables $F(x, w)$ such that $A(w)(x, u, \dot{u}) = F(x, w(x, u, \dot{u}))$ and equation (12) can be interpreted as the first-order ordinary differential equation

$$G_x + F(x, w) \cdot G_w = 0. \tag{13}$$

Suppose that $G = G(x, w)$ solves (13). Then by (10) $I(x, u, \dot{u}) = G(x, w(x, u, \dot{u}))$ satisfies $A(I) = 0$, i.e., $I(x, u, \dot{u})$ is a first integral of (1).

In summary, if we know that ∂_u is a λ-symmetry of (1) for some known function $\lambda(x, u, \dot{u})$, i.e., λ is a particular solution of (7), then a procedure to find a first integral $I(x, u, \dot{u})$ of (1), and consequently an integrating factor of that equation, reads as follows:

(i) Find a first integral $w(x, u, \dot{u})$ of $v^{[\lambda,(1)]}$, i.e., a particular solution of the equation

$$w_u + \lambda \cdot w_{\dot{u}} = 0. \tag{14}$$

(ii) Evaluate $A(w)$ and express $A(w)$ in terms of (x, w) as $A(w) = F(x, w)$.
(iii) Find a first integral G of $\partial_x + F(x, w)\partial_w$.
(iv) Evaluate $I(x, u, \dot{u}) = G(x, w(x, u, \dot{u}))$.

Then $I(x, u, \dot{u})$ is a first integral of (1) and $\mu(x, u, \dot{u}) = I_{\dot{u}}(x, u, \dot{u})$ is an integrating factor of (1).

3. First integrals, integrating factors and λ-symmetries: a first relationship

Theorem 2 and the former procedure work when $v = \partial_u$ is a λ-symmetry. If I is a first integral of (1), then $\mu = I_{\dot{u}}$ is an integrating factor of (1) and $-\mu\phi = I_x + \dot{u} I_u$. If I is also a first integral of $v^{[\lambda,(1)]}$ for some function $\lambda(x, u, \dot{u})$, then $I_u = -\lambda I_{\dot{u}} = -\lambda\mu$ and the system

$$I_x = \mu(\lambda\dot{u} - \phi), \qquad I_u = -\lambda\mu, \qquad I_{\dot{u}} = \mu, \tag{15}$$

is compatible; i.e., when $\lambda(x, u, \dot{u})$ is such that $v = \partial_u$ is a λ-symmetry, we know that system (15) is compatible. Therefore system (15) could be used to obtain through a line integral a first integral of (1) associated with μ.

Now we try to investigate the properties of two functions $\mu(x, u, \dot{u})$ and $\lambda(x, u, \dot{u})$ which make system (15) compatible. In this case what system (15) says is that there exists a common integral to the vector fields A and $v^{[\lambda,(1)]}$. We are going to prove that, when (15) is compatible, necessarily $v = \partial_u$ is a λ-symmetry.

First we observe that, if (15) is compatible,

$$\begin{aligned} \mu_x &= (I_{\dot{u}})_x = (I_x)_{\dot{u}} = (\mu(\lambda\dot{u} - \phi))_{\dot{u}} = \mu_{\dot{u}}(\lambda\dot{u} - \phi) + \mu(\lambda\dot{u} - \phi)_{\dot{u}}, \\ \mu_u &= (I_{\dot{u}})_u = (I_u)_{\dot{u}} = -\mu_{\dot{u}}\lambda - \mu\lambda_{\dot{u}}. \end{aligned} \tag{16}$$

By using (16) we obtain after some simplifications

$$\begin{aligned} I_{xu} &= -\mu_{\dot{u}}[\lambda(\lambda\dot{u} - \phi)] - \mu[\lambda_{\dot{u}}(\lambda\dot{u} - \phi) - (\lambda\dot{u} - \phi)_{\dot{u}}], \\ I_{ux} &= -\mu_{\dot{u}}[\lambda(\lambda\dot{u} - \phi)] - \mu[\lambda_x + \lambda(\lambda\dot{u} - \phi)_{\dot{u}}]. \end{aligned} \tag{17}$$

The compatibility of system (15) implies that $I_{xu} = I_{ux}$ and (17) gives us

$$\mu[-(\phi_u + \lambda\phi_{\dot{u}}) + (\lambda_x + \dot{u}\lambda_u + \phi\lambda_{\dot{u}} + \lambda^2)] = 0. \tag{18}$$

When $\mu \neq 0$, (18) implies that $v = \partial_u$ is a λ-symmetry. Hence we have proved the following theorem.

Theorem 3. *A system of the form (15) is compatible for some functions $\lambda(x, u, \dot{u})$ and $\mu(x, u, \dot{u})$ if and only if μ is an integrating factor of (1) and $v = \partial_u$ is a λ-symmetry of (1). In this case I is a first integral of (1).*

It must be observed that the compatibility of (15) implies that both μ and λ are uniquely defined by $\mu = I_{\dot{u}}$ and $\lambda = -I_u/I_{\dot{u}}$. In section 4 we prove that, when (λ, μ) makes (15) compatible, λ is also uniquely defined by μ as $\lambda = A(\mu)/\mu + \phi_{\dot{u}}$.

4. On integrating factors and λ-symmetries

In theorem 2 we have shown that if $v = \partial_u$ is a λ-symmetry of (1), there exists a common first integral of the vector fields $v^{[\lambda,(1)]}$ and A, i.e., the following system is compatible:

$$I_u + \lambda(x, u, \dot{u})I_{\dot{u}} = 0, \qquad I_x + \dot{u}I_u + \phi(x, u, \dot{u})I_{\dot{u}} = 0. \tag{19}$$

By differentiation with respect to $\dot{u}$ we obtain

$$I_{u\dot{u}} + \lambda_{\dot{u}}I_{\dot{u}} + \lambda I_{\dot{u}\dot{u}} = 0, \qquad I_{x\dot{u}} + I_u + \dot{u}I_{\dot{u}u} + \phi_{\dot{u}}I_{\dot{u}} + \phi I_{\dot{u}\dot{u}} = 0. \tag{20}$$

If we set $\mu = I_{\dot{u}}$, the first equation in (19) implies that $I_u = -\lambda\mu$ and system (20) can be written in terms of μ as

$$\mu_u + \lambda\mu_{\dot{u}} + \lambda_{\dot{u}}\mu = 0, \qquad A(\mu) + (\phi_{\dot{u}} - \lambda)\mu = 0. \tag{21}$$

Therefore, if $v = \partial_u$ is a λ-symmetry of (1), (21) is compatible and necessarily

$$\lambda = \frac{A(\mu)}{\mu} + \phi_{\dot{u}}. \tag{22}$$

The first equation in (21) can be written, without any dependence on λ, as

$$\mu_u + [A(\mu) + \mu\phi_{\dot u}]_{\dot u} = 0. \tag{23}$$

We now suppose that μ is any solution of (23) and that λ is defined by (22). Then

(i) μ is not necessarily an integrating factor of (1).
(ii) $v = \partial_u$ is not necessarily a λ-symmetry of (1).

This can be shown through the following example:

Example 4. Let us consider an equation of the form

$$\ddot u = \phi(x, u). \tag{24}$$

It is clear that, if $\mu = \mu(x)$, then μ is a solution of equation (23) and

$$\lambda = \frac{A(\mu)}{\mu} + \phi_{\dot u} = \frac{\dot\mu(x)}{\mu(x)}. \tag{25}$$

For most equations $\mu(x)$ is not an integrating factor of (24) and $\lambda(x) = \dot\mu(x)/\mu(x)$ is not a λ-symmetry of (24).

If one assumes that μ is a solution of (23) and λ is defined by (22), this raises the following questions:

(A) If λ is such that $v = \partial_u$ is a λ-symmetry of (1), is μ an integrating factor of (1)?
(B) If μ is an integrating factor of (1), is $v = \partial_u$ a λ-symmetry of (1)?

To answer question A it is enough to prove that, when μ is a solution of (23), the corresponding system (15) is compatible. By using (22), (23) and the third equation in (15)

$$(I_u)_{\dot u} = -(\lambda\mu)_{\dot u} = -[A(\mu) + \mu\phi_{\dot u}]_{\dot u} = \mu_u = (I_{\dot u})_u. \tag{26}$$

By using (26) we also have

$$\begin{aligned}(I_x)_{\dot u} &= [(\lambda\mu)_{\dot u} - \mu\phi]_{\dot u} = (\lambda\mu)_{\dot u}\dot u + (\lambda\mu) - \mu_{\dot u}\phi - \mu\phi_{\dot u}\\ &= -(I_u)_{\dot u}\dot u + (\lambda\mu) - \mu_{\dot u}\phi - \mu\phi_{\dot u}\\ &= -\mu_x - \dot u\mu_u - \phi\mu_{\dot u} - \mu\phi_{\dot u} + \mu_x + (\lambda\mu)\\ &= -[A(\mu) + \mu\phi_{\dot u}] + \mu_x + (\lambda\mu)\\ &= -[A(\mu) + \mu\phi_{\dot u}] + \mu_x + [A(\mu) + \mu\phi_{\dot u}] = \mu_x = (I_{\dot u})_x.\end{aligned} \tag{27}$$

It can be proven that $(I_x)_u$ and $(I_u)_x$ are given by (17). These second-order derivatives are identical because we assume that $v = \partial_u$ is a λ-symmetry of (1) and (18) holds.

Therefore the answer to question A is positive.

To answer question B we now suppose that μ is an integrating factor of (1). By theorem 1 we know that there exists a first integral J of (1) such that $\mu(\ddot u - \phi) = D_x J$. This equation implies that

$$J_{\dot u} = \mu, \tag{28}$$

$$-\mu\phi = J_x + \dot u J_u. \tag{29}$$

By differentiation of both members of (29) with respect to $\dot u$ and by using (28) we obtain

$$-\mu_{\dot u}\phi - \mu\phi_{\dot u} = J_{x\dot u} + J_u + \dot u J_{u\dot u} = \mu_x + \dot u\mu_u + J_u.$$

Therefore by (22),

$$J_u = -[\mu_x + \dot u\mu_u + \phi\mu_{\dot u} - \mu\phi_{\dot u}] = -[A(\mu) + \mu\phi_{\dot u}] = -\lambda\mu. \tag{30}$$

By using (28) and (30) we finally obtain that

$$J_x = -\mu\phi - \dot{u}J_u = -\mu\phi - \dot{u}(-\lambda\mu) = \mu(\lambda\dot{u} - \phi). \tag{31}$$

Equations (28), (30) and (31) prove that J is a solution of system (15) and this system is compatible. By theorem 3 $v = \partial_u$ is a λ-symmetry of (1). Thus we have proved the following theorem:

Theorem 5. *A function $\mu(x, u, \dot{u})$ is an integrating factor of (1) if and only if μ is a solution of (23) and $\lambda = A(\mu)/\mu + \phi_{\dot{u}}$ is such that $v = \partial_u$ is a λ-symmetry of (1).*

We now suppose that $\lambda(x, u, \dot{u})$ is such that $v = \partial_u$ is a λ-symmetry of (1). If $\mu(x, u, \dot{u})$ is such that $\lambda = A(\mu)/\mu + \phi_{\dot{u}}$, then theorem 5 ensures that, if μ satisfies (23), μ is an integrating factor of (1). Therefore we have obtained the following corollary.

Corollary 6. *If $\lambda(x, u, \dot{u})$ is such that $v = \partial_u$ is a λ-symmetry of (1) then any solution μ of the first-order linear system*

$$A(\mu) + (\phi_{\dot{u}} - \lambda)\mu = 0, \qquad \mu_u + (\lambda\mu)_{\dot{u}} = 0 \tag{32}$$

is an integrating factor of (1).

We now consider the relationship between integrating factors and Lie symmetries. Suppose that $v = \xi\partial_x + \eta\partial_u$ is a Lie point symmetry of (1). The first prolongation $v^{(1)}$ of v is $v^{(1)} = \xi\partial_x + \eta\partial_u + \eta^{(1)}\partial_{\dot{u}}$ where $\eta^{(1)} = D_x\eta - (D_x\xi)\dot{u}$. Since ξ and η do not depend upon $\dot{u}$, we can also write $\eta^{(1)} = A(\eta) - A(\xi)\dot{u} = A(\eta - \xi\dot{u}) + \xi\phi$. Therefore,

$$\begin{aligned} v^{(1)} &= \xi\partial_x + \eta\partial_u + A(\eta - \xi\dot{u})\partial_{\dot{u}} + \xi\phi\partial_{\dot{u}} \\ &= (\eta - \xi\dot{u})\partial_u + A(\eta - \xi\dot{u})\partial_{\dot{u}} + \xi[\partial_x + \dot{u}\partial_u + \phi\partial_{\dot{u}}] \\ &= v_Q^{(1)} + \xi \cdot A, \end{aligned} \tag{33}$$

where $Q = \eta - \xi\dot{u}$ is the characteristic of v and

$$v_Q^{(1)} = Q\partial_u + A(Q)\partial_{\dot{u}}. \tag{34}$$

Since v is a Lie point symmetry of (1), we have $[v^{(1)}, A] = -A(\xi)A$. By using (33) we can write

$$-A(\xi)A = [v^{(1)}, A] = \left[v_Q^{(1)} + \xi A, A\right] = \left[v_Q^{(1)}, A\right] - A(\xi)A. \tag{35}$$

Therefore $[v_Q^{(1)}, A] = 0$ and

$$\left[\left(\frac{1}{Q}v_Q^{(1)}\right), A\right] = -A\left(\frac{1}{Q}\right)v_Q^{(1)} = \left(\frac{A(Q)}{Q}\right)\cdot\left(\frac{1}{Q}v_Q^{(1)}\right) = \lambda\cdot\left(\frac{1}{Q}v_Q^{(1)}\right), \tag{36}$$

where $\lambda = A(Q)/Q$. If we denote $\bar{v} = \partial_u$, equation (36) proves that $\bar{v}$ is a λ-symmetry of (1). As a consequence of Corollary 6 we have the following theorem.

Theorem 7. *If v is a Lie point symmetry of (1) and Q is its characteristic, then $\bar{v} = \partial_u$ is a λ-symmetry of (1) for $\lambda = A(Q)/Q$ and any solution of the first-order linear system*

$$A(\mu) + \left(\phi_{\dot{u}} - \frac{A(Q)}{Q}\right)\mu = 0, \qquad \mu_u + \left(\frac{A(Q)}{Q}\mu\right)_{\dot{u}} = 0 \tag{37}$$

is an integrating factor of (1).

Example 8. Consider the second-order equation [37]

$$2u\ddot{u} - 6\dot{u}^2 + u^5 + u^2 = 0. \tag{38}$$

It is clear that $v = \partial_x$ is a Lie point symmetry of (38). It can be checked that this is the unique Lie point symmetry of (38).

The vector field associated with (38) is

$$A = \partial_x + \dot{u}\partial_u + \frac{1}{2}\left(\frac{6\dot{u}^2}{u} - u^4 - u\right)\partial_{\dot{u}} \tag{39}$$

and the characteristic of v is $Q = -\dot{u}$. The corresponding second equation of (37) becomes

$$\mu_u + \left(\frac{3}{u} - \frac{u^4+u}{2\dot{u}^2}\right)\dot{u}\mu_{\dot{u}} + \left(\frac{3}{u} + \frac{u^4+u}{2\dot{u}^2}\right)\mu = 0. \tag{40}$$

The solution of equation (40) can be obtained by the characteristic method of Lagrange:

$$\mu(x,u,\dot{u}) = \frac{2\dot{u}}{u^6}M\left(x, \frac{4\dot{u}^2 - 4u^5 - u^2}{4u^6}\right), \tag{41}$$

where M is an arbitrary function of x and $w = (4\dot{u}^2 - 4u^5 - u^2)/(4u^6)$. Since μ must also satisfy the first equation in (37),

$$M_x(x,w) = 0. \tag{42}$$

This implies that M depends only on w and

$$\mu(x,u,\dot{u}) = \frac{2\dot{u}}{u^6}M\left(\frac{4\dot{u}^2 - 4u^5 - u^2}{4u^6}\right) \tag{43}$$

is an integrating factor of (38), where M is an arbitrary function.

In order to find a first integral I of (38) such that $I_{\dot{u}} = \mu$, we must solve the system that corresponds to (15). If, for example, we choose $M(w) = w$ then we get the particular integrating factor $\mu(x,u,\dot{u}) = \dot{u}(4\dot{u}^2 - 4u^5 - u^2)/(2u^{12})$ and the corresponding system, (15), becomes

$$I_x = 0, \qquad I_u = -\frac{(u^5 + u^2 - 6\dot{u}^2)(4u^5 + u^2 - 4\dot{u}^2)}{4u^{13}}, \qquad I_{\dot{u}} = \frac{\dot{u}(4\dot{u}^2 - 4u^5 - u^2)}{2u^{12}}. \tag{44}$$

By evaluating the corresponding line integral we get the general solution of (44) and a class of first integrals of (38):

$$I(x,u,\dot{u}) = \frac{(4u^5 + u^2 - 4\dot{u}^2)^2}{32u^{12}} + C \qquad (C \in \mathbf{R}). \tag{45}$$

5. Integration by using two λ-symmetries

If $v_1 = \xi_1\partial_x + \eta_1\partial_u$, $v_2 = \xi_2\partial_x + \eta_2\partial_u$ are two vector fields on M and λ_1, λ_2 are two functions on $M^{(1)}$, then by (5),

$$\begin{aligned} v^{[\lambda_1,(1)]} &= \xi_1\partial_x + \eta_1\partial_u + [(A+\lambda_1)(Q_1) + \xi_1\phi]\partial_{\dot{u}}, \\ v^{[\lambda_2,(1)]} &= \xi_2\partial_x + \eta_2\partial_u + [(A+\lambda_2)(Q_2) + \xi_2\phi]\partial_{\dot{u}}, \end{aligned} \tag{46}$$

where $Q_i = \eta_i - \xi_i\dot{u}$ for $i = 1, 2$.

We investigate the linear dependence of the set of vector fields $\{A, v^{[\lambda_1,(1)]}, v^{[\lambda_2,(1)]}\}$. It can be checked that

$$\begin{vmatrix} 1 & \dot{u} & \phi \\ \xi_1 & \eta_1 & (A+\lambda_1)(Q_1) + \xi_1\phi \\ \xi_2 & \eta_2 & (A+\lambda_2)(Q_2) + \xi_2\phi \end{vmatrix} = Q_1(A+\lambda_2)(Q_2) - Q_2(A+\lambda_1)(Q_1). \tag{47}$$

This is a motivation to define an equivalence relationship between pairs of the form (v, λ). We say that two pairs (v_1, λ_1) and (v_2, λ_2) are A-equivalent and we write $(v_1, \lambda_1) \overset{A}{\sim} (v_2, \lambda_2)$ if and only if

$$Q_1(A + \lambda_2)(Q_2) - Q_2(A + \lambda_1)(Q_1) = 0. \tag{48}$$

It is clear that these two pairs are A-equivalent if and only if the set of vectors $\{A, v^{[\lambda_1,(1)]}, v^{[\lambda_2,(1)]}\}$ is linearly dependent. In this case we can write

$$v_1^{[\lambda_1,(1)]} = \frac{1}{Q_2}\left[\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} A + Q_1 v_2^{[\lambda_2,(1)]}\right]. \tag{49}$$

Suppose that v_1 is a λ_1-symmetry, v_2 is a λ_2-symmetry and $(v_1, \lambda_1) \overset{A}{\sim} (v_2, \lambda_2)$. If I is a first integral of vector fields A and $v_2^{[\lambda_2,(1)]}$, then by (49) I is also a first integral of $v_1^{[\lambda_1,(1)]}$.

We now prove that, if (v_1, λ_1) is not A-equivalent to (v_2, λ_2), there exists a first integral I_i of A and $v_i^{[\lambda_i,(1)]}$ that is not a first integral of $v_j^{[\lambda_j,(1)]}$ for $i, j \in \{1, 2\}$ and $i \neq j$. If there is a nonconstant function $I(x, u, \dot{u})$ such that $A(I) = 0$, $v_1^{[\lambda_1,(1)]}(I) = 0$ and $v_2^{[\lambda_2,(1)]}(I) = 0$, then this linear system would have a nontrivial solution and the determinant (47) would be identically null.

By using (49) it can be checked that for any pair (v_1, λ_1) the pair $(v_2, \lambda_2) = \left(\partial_u, \lambda_1 + \frac{A(Q_1)}{Q_1}\right)$ is A-equivalent to (v_1, λ_1). This proves that in any equivalence class there is a unique pair, (v, λ), such that $v = \partial_u$. In particular two pairs of the form (∂_u, λ_1) and (∂_u, λ_2) are A-equivalent if and only if $\lambda_1 = \lambda_2$. Therefore two different functions, λ_1 and λ_2, generate two different A-equivalence classes.

As a consequence of these results, if we manage to obtain two different solutions λ_1 and λ_2 of equation (7), then $v = \partial_u$ is both a λ_1-symmetry and a λ_2-symmetry. Two independent first integrals I_1 and I_2 of (1) can be obtained by solving the linear systems $\{A(I_1) = 0, v^{[\lambda_1,(1)]}(I_1) = 0\}$ and $\{A(I_2) = 0, v^{[\lambda_2,(1)]}(I_2) = 0\}$. This leads to the general solution of (1). In the next section we apply these results to the Ermakov–Pinney equation.

6. Applications to the Ermakov–Pinney equation

We consider the Ermakov–Pinney equation

$$\ddot{u} + \frac{b^2}{4u^3} + a(x)u = 0, \qquad b > 0, \tag{50}$$

where $a(x)$ is an arbitrary function depending upon x.

The vector field $v = \partial_u$ is a λ-symmetry of equation (50) if and only if $\lambda(x, u, \dot{u})$ is a solution of the determining equation,

$$\lambda_x + \dot{u}\lambda_u - \left(\frac{b^2}{4u^3} + a(x)u\right)\lambda_{\dot{u}} + \lambda^2 = \frac{3b^2}{4u^4} - a(x), \tag{51}$$

that corresponds to (7). In order to find some solutions of (51), we seek solutions that are linear in $\dot{u}$: $\lambda(x, u, \dot{u}) = \alpha(x, u)\dot{u} + \beta(x, u)$. Then α and β must satisfy

$$\alpha_x\dot{u} + \beta_x + \dot{u}(\alpha_u\dot{u} + \beta_u) - \left(\frac{b^2}{4u^3} + a(x)u\right)\alpha + (\alpha\dot{u} + \beta)^2 - \frac{3b^2}{4u^4} + a(x) = 0. \tag{52}$$

It is clear that the first member of (52) is a second-order polynomial in $\dot{u}$ the coefficients of which must be null. Therefore

$$\alpha_u + \alpha^2 = 0, \tag{53}$$

$$\alpha_x + \beta_u + 2\alpha\beta = 0, \tag{54}$$

$$\beta_x - \left(\frac{b^2}{4u^3} + a(x)u\right)\alpha + \beta^2 - \frac{3b^2}{4u^4} + a(x) = 0. \tag{55}$$

It is clear that $\alpha(x, u) = 1/u$ is a particular solution of (53). For this α equation (54) becomes

$$\beta_u + 2\frac{1}{u}\beta = 0. \tag{56}$$

The general solution of (56) is of the form

$$\beta(x, u) = \frac{r(x)}{u^2}, \tag{57}$$

where $r(x)$ is an arbitrary function on x. This function β must satisfy (55) and therefore $r(x)$ must satisfy the equation:

$$\frac{r'(x)}{u^2} - \left(\frac{b^2}{4u^4} + a(x)\right) + \frac{r(x)^2}{u^4} - \frac{3b^2}{4u^4} + a(x) = 0. \tag{58}$$

It is clear that $r_1(x) = b$ and $r_2(x) = -b$ are two different solutions of (58) which determine the two following solutions of (51):

$$\lambda_1(x, u, \dot{u}) = \frac{1}{u}\dot{u} + \frac{b}{u^2}, \qquad \lambda_2(x, u, \dot{u}) = \frac{1}{u}\dot{u} - \frac{b}{u^2}. \tag{59}$$

Hence $v = \partial_u$ is both a λ_1-symmetry and a λ_2-symmetry of (50). Since $\lambda_1 \neq \lambda_2$, the pairs (v, λ_1) and (v, λ_2) are not A-equivalent. It can be checked that in any of the two equivalence classes associated with these two pairs there are no Lie point symmetries.

We now calculate a first integral $I(x, u, \dot{u})$ associated with the λ_1-symmetry $v = \partial_u$. This requires one to solve the following system:

$$I_x + \dot{u}\phi_u - \left(\frac{b^2}{4u^4} + a(x)\right) I_{\dot{u}} = 0, \tag{60}$$

$$I_u + \left(\frac{\dot{u}}{u} + \frac{b}{u^2}\right) I_{\dot{u}} = 0, \tag{61}$$

that corresponds to (19).

The general solution of equation (61) is given by

$$I(x, u, \dot{u}) = H\left(x, \frac{\dot{u}}{u} + \frac{b}{2u^2}\right), \tag{62}$$

where H is an arbitrary function of x and $w = \dot{u}/u + b/(2u^2)$. Since I must also satisfy (60), H satisfies the equation

$$H_x(x, w) + (w^2 - a(x))H_w(x, w) = 0. \tag{63}$$

We assume that $w_0 = w_0(x)$ is a known particular solution of the Riccati equation

$$\dot{w} + w^2 + a(x) = 0. \tag{64}$$

If $w_1(x)$ is such that $\dot{w}_1 = -2w_0w_1$ and $\dot{w}_2 = -w_1$, then

$$H(x, w) = \frac{w_1(x)}{w - w_0(x)} + w_2(x) \tag{65}$$

is a solution of (63). Therefore a first integral of equation (50) is given by

$$I_1(x, u, \dot{u}) = \frac{2w_1(x)u^2}{2u\dot{u} + b - 2u^2w_0(x)} + w_2(x). \tag{66}$$

An integrating factor associated with I_1 is given by

$$\mu_1(x,u,\dot{u}) = -\frac{4w_1(x)u^3}{(2u\dot{u}+b-2u^2w_0(x))^2}. \tag{67}$$

A similar procedure can be used to calculate a first integral associated with the λ_2-symmetry $v = \partial u$. In this case we would obtain

$$I_2(x,u,\dot{u}) = \frac{2w_1(x)u^2}{2u\dot{u}-b-2u^2w_0(x)} + w_2(x). \tag{68}$$

The corresponding integrating factor is

$$\mu_2(x,u,\dot{u}) = -\frac{4w_1(x)u^3}{(2u\dot{u}-b-2u^2w_0(x))^2}. \tag{69}$$

Since $\lambda_1 \neq \lambda_2$, I_1 and I_2 are functionally independent first integrals of (50).

The general solution of (50) can be obtained by eliminating u from the system

$$I_1(x,u,\dot{u}) = C_1, \qquad I_2(x,u,\dot{u}) = C_2, \qquad (C_1, C_2 \in \mathbb{R}). \tag{70}$$

This implies that the solutions of (50) must satisfy

$$u^2(C_1 - C_2)w_1(x) + b(C_1 - w_2(x))(C_2 - w_2(x)) = 0. \tag{71}$$

7. On λ-symmetries and other methods of construction of first integrals

7.1. The Prelle–Singer method

Prelle and Singer [34] introduced a method to construct integrating factors of first-order ODEs. This method has been adapted and applied to second-order ODEs in [35] and has been extended to nth-order ODEs in [36]. The main characteristics read as follows. They consider the equation

$$\ddot{u} = \frac{P(x,u,\dot{u})}{Q(x,u,\dot{u})}, \qquad P, Q \in \mathbf{C}[x,u,\dot{u}], \tag{72}$$

and the associated differential form

$$\frac{P}{Q}\,\mathrm{d}x - \mathrm{d}\dot{u}. \tag{73}$$

By adding a differential form of type $S(x,u,\dot{u})(\dot{u}\,\mathrm{d}x - \mathrm{d}u)$ to the differential form (73), where S is an unknown function, they consider the differential form

$$\left(\frac{P}{Q} + S\dot{u}\right)\mathrm{d}x - [S\mathrm{d}u + \mathrm{d}\dot{u}]. \tag{74}$$

The extended Prelle–Singer method tries to find a function S such that the differential form (74) is proportional to the differential form $\mathrm{d}I = I_x\,\mathrm{d}x + I_u\,\mathrm{d}u + I_{\dot{u}}\,\mathrm{d}\dot{u}$, for some function $I(x,u,\dot{u})$. This means that there exists some function R such that

$$\mathrm{d}I = R\left[\left(\frac{P}{Q} + S\dot{u}\right)\mathrm{d}x - (S\mathrm{d}u + \mathrm{d}\dot{u})\right]. \tag{75}$$

The existence of the functions S, I and R satisfying (75) implies that

$$I_x = R\left(\frac{P}{Q} + \dot{u}S\right), \qquad I_u = -RS, \qquad I_{\dot{u}} = -R. \tag{76}$$

The compatibility conditions for system (76) imply

$$A(S) = -\phi_u + S\phi_{\dot{u}} + S^2, \qquad A(R) = -R(S+\phi_{\dot{u}}), \qquad R_x = R_{\dot{u}}S + RS_{\dot{u}}, \tag{77}$$

where $\phi = P/Q$ and A is the operator associated with equation (73). The first equation in (77) says that $v = \partial_u$ is a λ-symmetry for $\lambda = -S$. By writing the second and third equations of (77) in terms of λ we obtain

$$R_x + \dot{u} R_u + \phi R_{\dot{u}} = R(\lambda - \phi_{\dot{u}}), \qquad R_u + \lambda R_{\dot{u}} + R\lambda_{\dot{u}} = 0. \tag{78}$$

When one sets $\mu = -R$, this system is equivalent to system (21). If λ and μ (or R and S) are known, then the system (76) is equivalent to system (15).

This reveals a role of $\lambda = -S$ on the integration of equation (73): if we add $-\lambda(\dot{u}\,\mathrm{d}x - \mathrm{d}u)$ to the differential form $\phi \mathrm{d}x - \mathrm{d}\dot{u}$ (i.e., the differential form (73)), then the resulting differential form

$$\phi \mathrm{d}x - \mathrm{d}\dot{u} - \lambda(\dot{u}\,\mathrm{d}x - \mathrm{d}u)$$

admits an integrating factor μ.

In [36] several interesting examples are considered to apply the generalized Prelle–Singer method. It must be mentioned that the functions S that correspond to these examples follow from Lie point symmetries of the equations, i.e., $S = -A(Q)/Q$, where Q is the characteristic of some Lie point symmetry of the equation:

(1) For the harmonic oscillator equation,

$$\ddot{u} = -u, \tag{79}$$

in [36] the function $S_1 = u/\dot{u}$ is considered. This function corresponds to the Lie point symmetry $v_1 = \partial_x$. A second function S_2, which leads to an independent first integral of (79), is given by $S_2 = \tan x$. This second function corresponds to the Lie point symmetry $v_2 = \cos x \partial_u$. The two Lie symmetries v_1 and v_2 are in different equivalence classes because

$$Q_1 A(Q_2) - Q_2 A(Q_1) = Q_1 Q_2(-S_2 + S_1) \neq 0. \tag{80}$$

(2) In the study of a relativistic fluid sphere, Buchdahl [38] obtained the equation

$$u\ddot{u} - 3\dot{u}^2 - x^{-1} u\dot{u} = 0. \tag{81}$$

This equation admits the Lie point symmetry $v_1 = (1/x)\partial_x$. The corresponding function S_1 is

$$S_1 = -\frac{A(Q_1)}{Q_1} = -\frac{3\dot{u}}{u}, \tag{82}$$

which also appears in [36]. Other Lie point symmetries of equation (81) are $v_2 = x\partial_x$ and $v_3 = (1/(u^2 x))\partial_x$. The corresponding characteristics are $Q_2 = -x\dot{u}$ and $Q_3 = -\dot{u}/(u^2 x)$. For these characteristics we have

$$\frac{A(Q_2)}{Q_2} = \frac{3\dot{u}}{u} + \frac{2}{x} = -S_2. \tag{83}$$

$$\frac{A(Q_3)}{Q_3} = \frac{\dot{u}}{u} = -S_3. \tag{84}$$

It should be mentioned that function S_3 was obtained in [39] by using ad hoc methods and not as a consequence of a general procedure. It can be checked that the pairs $(v_1, -S_1)$, $(v_2, -S_2)$ and $(v_3, -S_3)$ are in different equivalence classes and therefore any two of these three pairs can be used to obtain the general solution of (81).

7.2. The variational derivative and integrating factors

The classical approach for obtaining the determining equations for integrating factors of equation (1) is, in short, as follows. If μ is an integrating factor of (1), then the function $\theta(x, u, \dot{u}) = \mu(x, u, \dot{u})(\ddot{u} - \phi(x, u, \dot{u}))$ must be a total derivative and therefore its variational derivative is null, i.e., θ is an invariant of the (truncated) Euler operator

$$E = \partial_u - D_x\partial_{\dot{u}} + D_x^2\partial_{\ddot{u}}. \tag{85}$$

Since we can write

$$D_x = A + (\ddot{u} - \phi)\partial_{\dot{u}}, \tag{86}$$

it is easy to check that

$$E(\theta) = \ddot{u}[\mu_u + [A(\mu) + \mu\phi_{\dot{u}}]_{\dot{u}}] + [A[A(\mu) + \mu\phi_{\dot{u}}] - \mu\phi_u - \phi(\mu_u + [A(\mu) + \mu\phi_{\dot{u}}]_{\dot{u}})]. \tag{87}$$

The classical necessary and sufficient conditions for μ be an integrating factor of (1) are the equations

$$\mu_u + [A(\mu) + \mu\phi_{\dot{u}}]_{\dot{u}} = 0 \tag{88}$$

$$A[A(\mu) + \mu\phi_{\dot{u}}] - \mu\phi_u - \phi(\mu_u + [A(\mu) + \mu\phi_{\dot{u}}]_{\dot{u}}) = 0. \tag{89}$$

These are the determining equations that appear, for instance, in [1] or in [2].

Equations (88) and (89) can directly be obtained without the use of variational derivatives as a consequence of compatibility conditions. If we write system (15) in terms of I and μ by using $\lambda = [A(\mu) + \mu\phi_{\dot{u}}]/\mu$, we obtain the system

$$I_x = [A(\mu) + \mu\phi_{\dot{u}}]\dot{u} - \phi\mu, \qquad I_u = -[A(\mu) + \mu\phi_{\dot{u}}], \qquad I_{\dot{u}} = \mu. \tag{90}$$

The compatibility condition $(I_{\dot{u}})_u = (I_u)_{\dot{u}}$ gives

$$(A(\mu) + \mu\phi_{\dot{u}})_{\dot{u}} + \mu_u = 0. \tag{91}$$

This equation is the same as (23) and (88).

If we use the compatibility condition $(I_u)_x = (I_x)_u$, we obtain

$$[A(\mu) + \mu\phi_{\dot{u}}]_u\dot{u} - \phi_u\mu - \phi\mu_u + [A(\mu) + \mu\phi_{\dot{u}}]_x = 0. \tag{92}$$

By adding $\phi[A(\mu) + \mu\phi_{\dot{u}}]_{\dot{u}}$ to both sides of (92) we obtain

$$A[A(\mu) + \mu\phi_{\dot{u}}] - \mu\phi_u - \phi[(A(\mu) + \mu\phi_{\dot{u}})_{\dot{u}} + \mu_u] = 0. \tag{93}$$

This is equation (89). By (91) the coefficient of ϕ in equation (93) is null. Therefore system (88) and (89) is equivalent to the system

$$\mu_u + [A(\mu) + \mu\phi_{\dot{u}}]_{\dot{u}} = 0, \qquad A[A(\mu) + \mu\phi_{\dot{u}}] - \mu\phi_u = 0. \tag{94}$$

It must observed that for obtaining these equations we have not used, explicitly, variational principles.

We now show the role of λ-symmetries in this context. If we set $\lambda\mu = A(\mu) + \mu\phi_{\dot{u}}$, then $A(\mu) = (\lambda - \phi_{\dot{u}})\mu$ and

$$A[A(\mu) + \mu\phi_{\dot{u}}] = A(\lambda\mu) = A(\lambda)\mu + \lambda A(\mu) = [A(\lambda) + \lambda^2 - \lambda\phi_{\dot{u}}]\mu. \tag{95}$$

Therefore in terms of μ and $\lambda = A(\mu)/\mu + \phi_{\dot{u}}$,

$$E(\theta) = (\ddot{u} - \phi)[\mu_u + [A(\mu) + \mu\phi_{\dot{u}}]_{\dot{u}}] + \mu[A(\lambda) + \lambda^2 - \lambda\phi_{\dot{u}} - \phi_u]. \tag{96}$$

It must be observed that the determining equation (7) for λ-symmetries is $A(\lambda)+\lambda^2-\lambda\phi_{\dot{u}}-\phi_u = 0$, the first member of which is the coefficient of μ in equation (96).

Theorem 5 could directly be derived from (96). In fact, if μ is an integrating factor of (1), then $E(\theta) = 0$ and therefore μ satisfies the first equation in (94) and (96) implies that $v = \partial_u$ is a λ-symmetry of (1) for $\lambda = A(\mu)/\mu + \phi_{\dot{u}}$. Conversely, if μ satisfies the first equation in (94) and ∂_u is a λ-symmetry, then (96) implies that μ is an integrating factor of (1).

Our next example illustrates the way λ-symmetries help to find integrating factors of an equation.

7.2.1. Painlevé XIV equation

$$\ddot{u} - \frac{\dot{u}^2}{x} + \dot{u}\left(-uq(x) - \frac{s(x)}{u}\right) + s'(x) - q'(x)u^2 = 0. \tag{97}$$

It can be checked that equation (97) has no Lie point symmetries and hence cannot be integrated by the Lie method of reduction.

To calculate an integrating factor by the method based on variational derivatives one has to find particular solutions of the following system of determining equations

$$2\mu + 2\mu_u + 2\left(qu^2 + 2\dot{u} + s\right)\mu_{\dot{u}} + u\mu_{x\dot{u}} + u\dot{u}\mu_{u\dot{u}} + (q'u^3 + \dot{u}qu^2 - s'u + \dot{u}^2 + \dot{u}s)\mu_{\dot{u}\dot{u}} = 0, \tag{98}$$

$$\begin{aligned} &-\mu(q'u^3 - s'u + \dot{u}^2) + (qu^3 + 2u\dot{u} + su)\mu_x + (\dot{u}^2u - q'u^4 + s'u^2)\mu_u \\ &\quad + (q''u^4 + 3\dot{u}q'u^3 + \dot{u}^2qu^2 - s''u^2 + \dot{u}s'u - \dot{u}^3 - \dot{u}^2s)\mu_{\dot{u}} \\ &\quad + u^2\mu_{xx} + 2u^2\dot{u}\mu_{xu} + (q'u^4 + \dot{u}qu^3 - s'u^2 + \dot{u}^2u + \dot{u}su)\mu_{x\dot{u}} \\ &\quad + \dot{u}u(\dot{u}u + q'u^3 + \dot{u}qu^2 - s'u + \dot{u}^2 + \dot{u}s)\mu_{u\dot{u}} = 0. \end{aligned} \tag{99}$$

Ibragimov ([1], p 241) has considered a particular case of this equation ($q(t) = 0$, $s(t) = t^2 + t$) and has found an integrating factor of the equation by using an specific *ansatz* to solve system (98) and (99). This particular case also appears in [7]. Nucci [40] has used a method inspired by the Jacobi last multiplier for another particular case ($q(t) = 1$, $s(t) = t$) to find a first integral of Riccati type.

To obtain the general solution of system (98) and (99) by standard methods seems a quite difficult task. We now try to seek a λ-symmetry of the equation. Since ϕ is a quadratic polynomial in $\dot{u}$, the special form of the determining equation (7) suggests to seek particular solutions of the form $\lambda(x, u, \dot{u}) = \alpha(x, u)\dot{u} + \beta(x, u)$. The corresponding determining equations for α and β are

$$(\alpha^2 + \alpha_u)u^2 - \alpha u + 1 = 0, \tag{100}$$

$$(2\alpha\beta - q + \beta_u + \alpha_x)u^2 - 2\beta u + s = 0, \tag{101}$$

$$\alpha q'u^3 - (\beta q + 2q')u^2 + (\beta^2 - \alpha s' + \beta_x)u - \beta s = 0. \tag{102}$$

It is clear that $\alpha(x, u) = \frac{1}{u}$ is a particular solution of equation (100). Equation (101) becomes

$$(\beta_u - q)\,u^2 + s = 0, \tag{103}$$

the general solution of which is of the form $\beta(x, u) = uq(x) + \frac{s(x)}{u} + c_1(x)$, for some arbitrary function $c_1(x)$. It is easy to check that equation (102) is satisfied for $c_1(x) = 0$. Hence $v = \partial_u$ is a λ-symmetry of (97) for $\lambda = \frac{\dot{u}}{u} + uq(x) + \frac{s(x)}{u}$. By corollary 6 any particular solution μ of the first-order linear system of PDEs

$$\begin{aligned} &\dot{u}\mu + (q'u^3 + \dot{u}qu^2 - s'u + \dot{u}^2 + \dot{u}s)\mu_{\dot{u}} + u(\dot{u}\mu_u + \mu_x) = 0, \\ &\mu + (qu^2 + \dot{u} + s)\mu_{\dot{u}} + u\mu_u = 0 \end{aligned} \tag{104}$$

is an integrating factor of (97). The general solution of equation (104) can be obtained by the characteristic method of the Lagrange: a family of integrating factors of (97) is given by

$$\mu(x, u, \dot{u}) = \frac{1}{u}M\left(\frac{\dot{u} - q(x)u^2 + s(x)}{u}\right). \tag{105}$$

8. Conclusions

This paper explores the use of λ-symmetries in the investigation of the first integrals and integrating factors of a given second-order ODE. When a λ-symmetry of the equation is known, several methods are presented to obtain the associated first integrals or integrating factors; in particular the associated integrating factors can be found by means of a coupled first-order linear system of PDEs, instead of a second-order linear system of PDEs. When two nonequivalent λ-symmetries of the equation are known, an algorithm to find two functionally independent first integrals of the equation is presented. In this paper other known methods to find first integrals and integrating factors are considered: methods based on variational derivatives or extensions of the Prelle–Singer method. The results considered in this paper cover and complete these known methods. Several applications to ODEs that appear in the investigation of relevant equations of mathematical physics are also considered.

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